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Units from 5-Torsion on the Jacobian of $y^2 = x^5 + 1/4$ and the Conjectures of Stark and Rubin

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expansion of the Artin L-series $L(s, \chi)$ at $s=0$ to the determinant of a matrix whose entries are linear combinations of logs of absolute values of units in K [StI–IV, Tat1]. In the case that G is abelian and the L-series $L(s, \chi)$ has a zero of order one at $s=0$, Stark gave a refined conjecture for the precise value of $L'(0, \chi)$. Stark proved this refined conjecture in the case that $k = \mathbb{Q}$ or k is an imaginary quadratic field, making use of cyclotomic and elliptic units [StIV]. Only scant progress has been made in generalizing elliptic units to "abelian units" attached to abelian varieties of dimension greater than 1 [BaBo, BoBa, dSG, Gra2, A].

Recently Rubin gave a generalized refined Stark's conjecture for the value of the lead term of $L(s, \chi)$ at $s=0$ whenever G is abelian [R]. In this paper we describe a set of abelian units attached to the 5-torsion of the Jacobian of the curve $y^2 = x^5 + 1/4$ which can be used to verify Rubin's conjecture in a case where the L-series has a second order zero at $s=0$. Stark also questioned what the lead term should be when there is a second order zero [St2]. These units can also be used to affirm his question in this case.

In the first section of the paper we recall the various conjectures. In the second section we describe the fields k and K of our example. In Section 3 we discuss the appropriate units, and in Section 4 we describe how to numerically evaluate the L-series. In the fifth section we establish a key equality, and in the last section we explain for this example how to derive the conjectures from the key equality. Finally, the appendix contains a description of the geometry behind the construction of the units.

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A note on calculations. There are many calculations in the paper. They were all made using the program *Mathematica*. We will only equate a quantity Y and a rounded computed value Z if Y is known to be an integer and Z is an integer to 10 significant digits. We will write $Y \approx Z$ if Z is a truncated computed value of Y which seems to be correct to 10 significant digits. I have not done the numerical analysis to guarantee that the calculations are correct to 10 significant digits—I have just done the *Mathematica* computations using its standard internal precision (16 significant digits on my machine) and trust that they are correct to 10 significant digits.

1. THE CONJECTURES

Let K/k be an abelian extension of numbers fields. Let S be a finite set of places of k containing the archimedean places and all the places which ramify in K. Let G be the Galois group of K/k and χ a character on G. Then the Artin L-series $L_S(s, \chi)$ is defined for $\mathrm{Re}(s) > 1$ by

$$L_S(s, \chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathrm{Fr}_{\mathfrak{p}})\, N\mathfrak{p}^{-s})^{-1},$$

where the $\mathfrak{p}$ are primes ideals in the integers of k, $\mathrm{Fr}_{\mathfrak{p}}$ is the Frobenius in G attached to $\mathfrak{p}$, and N denotes the absolute norm. If $r(\chi)$ denotes the order of vanishing of $L_S(s, \chi)$ at $s = 0$, then $r(\chi) = \#(S) - 1$ if χ is the trivial character χ_1, and otherwise is the number of primes $\mathfrak{q}$ in S such that χ restricted to the decomposition group of $\mathfrak{q}$ is trivial.

From now on we will assume that S is a finite set of places of k containing the archimedean places, all the places which ramify in K/k, a distinguished set of r primes $\mathfrak{p}_1, ..., \mathfrak{p}_r$ which split completely in K/k, and is such that $\#(S) > r$. Then we will always have $r(\chi) \geqslant r$. Let W_K denote the number of roots of unity of K. For a place $\mathfrak{P}$ in K we let $|\alpha|_{\mathfrak{P}}$ denote the standard absolute value $\pm\alpha$ if $\mathfrak{P}$ is real, α times its complex conjugate if $\mathfrak{P}$ is complex, and $N\mathfrak{P}^{-ord_{\mathfrak{P}}(\alpha)}$ if $\mathfrak{P}$ is finite. In the case that $r = 1$, Stark gave a refined conjecture for the value of $L'(0, \chi)$ [StI–IV]. The following is the version in [Tat1].

CONJECTURE (Stark). *Let $\mathfrak{P}_1$ be any prime of K sitting over the distinguished prime $\mathfrak{p}_1$ in S. Then there is an $\varepsilon_1 \in K$ which is a unit at all places of K not above S, such that*

(a) *for all characters χ on G,*

$$L'(0, \chi) = -\frac{1}{W_K} \sum_{\gamma \in G} \chi(\gamma) \log |\varepsilon_1^{\gamma}|_{\mathfrak{P}_1},$$

and

(b) $K(\varepsilon_1^{1/W_K})$ *is an abelian extension of* k.

In addition, if $\#(S)>2$, ε_1 *is a unit outside places dividing* $\mathfrak{p}_1$, *and if* $S=\{\mathfrak{p}_1, \mathfrak{q}\}$, *then for all places* $\mathfrak{Q}$ *of* K *over* $\mathfrak{q}$, $|\varepsilon_1^\gamma|_{\mathfrak{Q}}$ *is independent of the choice of* $\gamma \in G$.

This conjecture was proved by Stark when k is $\mathbb{Q}$ or k is an imaginary quadratic field, and by Sands [Sa1, Sa2, Sa3] and Tate [Tat1] in certain other cases (see [DST] for a resume of known results). If $\mathfrak{p}_1$ is an archimedean place, and one can compute the L-series, then the conjecture can be numerically verified by assuming its truth, thereby determining the absolute values of ε_1, and therefore bounding the coefficients of a polynomial with root ε_1. One can sometimes do even better: see [St1] and [DST] for some computations.

In a talk in 1980, Stark wondered what the lead term of $L_S(s,\chi)$ should look like when $r=2$. Although he never stated it as a conjecture, he questioned whether the following should hold [St2]:

QUESTION (Stark).[1] *If* $r=2$, *given primes* $\mathfrak{P}_i$ *of* K *lying over* $\mathfrak{p}_i$, $i=1,2$, *do there exist* S*-units* ε_1 *and* ε_2 *in* K *such that*

(a)

$$\frac{1}{2!}L_S^{(2)}(0,\chi)=\operatorname{Det}\left[\sum_{\gamma\in G}\chi(\gamma)\begin{pmatrix}-\dfrac{1}{W_K}\log|\varepsilon_1^\gamma|_{\mathfrak{P}_1} & -\dfrac{1}{W_K}\log|\varepsilon_1^\gamma|_{\mathfrak{P}_2}\\ -\dfrac{1}{W_K}\log|\varepsilon_2^\gamma|_{\mathfrak{P}_1} & -\dfrac{1}{W_K}\log|\varepsilon_2^\gamma|_{\mathfrak{P}_2}\end{pmatrix}\right],$$

and

(b) $K(\varepsilon_1^{1/W_K})$ *and* $K(\varepsilon_2^{1/W_K})$ *are abelian over* k?

Follow-up Question. Let $\mathcal{O}_K$ *be the ring of integers of* K. *Can one demand that* $K(\varepsilon_1^{1/W_K})=K(\varepsilon_2^{1/W_K})$, *and that for* $i=1,2$, $\varepsilon_i^\gamma\,\mathcal{O}_K=\varepsilon_i\,\mathcal{O}_K$ *for all* $\gamma\in G$?

Although the question will be answered in the affirmative for the example we compute in this paper, and Tangedal [Tan] has affirmed the question in several cases where K/k is quadratic, there may be reason to doubt that (a) always holds. Indeed, Rubin [R] has shown

[1] Stark originally asked the question only in the case that k is real quadratic, and refined it in [Tan], but said in a recent conversation that it should be asked in this more general context.

that for general r, there are not always S-units $\varepsilon_1, ..., \varepsilon_r$ in K such that $(1/r!)\, L_S^{(r)}(0, \chi)\, W_k^r$ is

$$\operatorname{Det}\left(\sum_{\gamma \in G} \chi(\gamma)[\log |\varepsilon_i^\gamma|_{\mathfrak{P}_j}]_{i,j}\right), \tag{1}$$

(where $\mathfrak{P}_j$ lies over $\mathfrak{p}_j$). This provided a counterexample to Sands's conjecture for the value of $L_S^{(r)}(0, \chi)$ [Sa4]. In any case, Stark's formalism turns out to be a good place to start for our example.

Recently Rubin gave a refined conjecture for the value of $L_S^{(r)}(0, \chi)$ whenever G is abelian [R]. He expresses the lead term of an L-series as a rational linear combination of determinants of the type in (1), whose coefficients are allowed, in a carefully prescribed manner, to have primes in their denominator that divide the order of G. To give this precisely, we need some notation from [R].

Let S be as above, and let T be a set of primes of k disjoint from S. Then define

$$L_{S,T}(s, \chi) = L_S(s, \chi) \prod_{\mathfrak{p} \in T} (1 - \chi(\mathrm{Fr}_{\mathfrak{p}})\, N\mathfrak{p}^{1-s}).$$

For each $\chi \in \hat{G}$ define $e_\chi = (1/|G|) \sum_{\gamma \in G} \chi(\gamma)\, \gamma^{-1}$ in the group ring $\mathbb{C}[G]$, and set

$$\theta_{S,T}^{(r)}(0) = \sum_{\chi \in \hat{G}} e_\chi \frac{1}{r!}\, L_{S,T}^{(r)}(0, \bar{\chi}).$$

For any $\mathbb{Z}[G]$-module $\mathcal{M}$, and any natural number r, let $\bigwedge^r \mathcal{M}$ denote the rth-exterior product of $\mathcal{M}$. For any $\varphi_1, ..., \varphi_r$ in $\operatorname{Hom}(\mathcal{M}, \mathbb{Z}[G])$, we get a $\mathbb{Z}[G]$-homomorphism $\varphi_1 \wedge \cdots \wedge \varphi_r$ from $\bigwedge^r \mathcal{M}$ into $\mathbb{Z}[G]$, defined on generators $m = m_1 \wedge \cdots \wedge m_r$ by

$$\varphi_1 \wedge \cdots \wedge \varphi_r(m) = \operatorname{Det}([\varphi_i(m_j)]_{i,j}).$$

Let $\bigwedge_0^r \mathcal{M}$ denote the submodule of $\mathbb{Q} \otimes \bigwedge^r \mathcal{M}$ consisting of the elements m such that for all $\varphi_i \in \operatorname{Hom}(\mathcal{M}, \mathbb{Z}[G])$,

$$\varphi_1 \wedge \cdots \wedge \varphi_r(m) \in \mathbb{Z}[G].$$

For any vector $\eta = (\mathfrak{P}_1, ..., \mathfrak{P}_r)$ of r places of K, define a regulator map $R_\eta : \bigwedge^r K^* \to \mathbb{R}[G]$ by setting

$$R_\eta(u_1 \wedge \cdots \wedge u_r) = \operatorname{Det}\left(\left[\sum_{\gamma \in G} \log |u_i^\gamma|_{\mathfrak{P}_j}\, \gamma^{-1}\right]_{i,j}\right).$$

Let $U_{S,T}$ be the set of numbers in K which are units at every prime not dividing a prime in S, and which are congruent to 1 modulo every prime in K above a prime in T. Then $U_{S,T}$ is a $\mathbb{Z}[G]$-module. Hence we have a map $R\eta : \bigwedge^r U_{S,T} \to \mathbb{R}[G]$ which we extend to $\bigwedge_0^r U_{S,T}$ by linearity.

Finally, define $\Lambda_{S,T}$ to be the submodule of $\bigwedge_0^r U_{S,T}$ containing the elements α, such that $e_\chi \alpha = 0$ in $\mathbb{C} \otimes \bigwedge^r U_{S,T}$ for every $\chi \in \hat{G}$ such that $r(\chi) > r$.

CONJECTURE (Rubin) [R]. *Let S be as above, and let T be chosen so that $U_{S,T}$ is torsion-free. Let $\mathfrak{P}_i$ be a prime of K lying over $\mathfrak{p}_i$, and $\eta = (\mathfrak{P}_1, ..., \mathfrak{P}_r)$. Then there exists a unique $\varepsilon_{S,T} \in \Lambda_{S,T}$ such that*

$$R_\eta(\varepsilon_{S,T}) = \theta_{S,T}^{(r)}(0).$$

Rubin has shown the uniqueness of $\varepsilon_{S,T}$. He has proved the conjecture for the cases when S contains more than r places which split completely, for quadratic extensions, and when $r = 0$. When $r = 1$, his conjecture for fixed k, K, S, and all appropriate T, is equivalent to Stark's conjecture for k, K, and S [R].

It is difficult to numerically verify Rubin's Conjecture for $r > 1$ or Stark's question, because even knowing the values of the L-series, one cannot determine the absolute values of the units involved. Seemingly, one needs to know a lot about the units of K to begin with. The author was in such a happy circumstance after working on [Gra3].

The purpose of this paper is to verify Rubin's Conjecture and Stark's question for a fixed choice of k and K arising from the arithmetic of a curve of genus 2.

2. THE FIELDS k AND K

Let ζ denote a primitive fifth root of unity, let $k = \mathbb{Q}(\zeta)$, a cyclic quartic extension of $\mathbb{Q}$, and let $\mathcal{O}_k = \mathbb{Z}[\zeta]$ be its ring of integers. We let $\varepsilon = -\zeta^2 - \zeta^3$. Then $2\varepsilon - 1$ is a square root of 5 which we denote by $\sqrt{5}$. Also ε is a fundamental unit for the real quadratic subfield of k, and hence for k. Therefore the regulator R_k of k is $|2 \log |\varepsilon||$, where $|\,|$ denotes the real absolute value. It is well known that k has class number $h_k = 1$, has $W_k = 10$ roots of unity, and has discriminant $D_k = 5^3$. We let $\lambda = 1 - \zeta$, which generates the lone prime above 5 in k.

It was shown in [Gra3] that every $\alpha \in \mathcal{O}_k$ prime to λ has an associate $\alpha' \equiv 1 \bmod \lambda^3$ (if α is real, the congruence must hold mod λ^4). Hence k is its own ray class field of conductor λ^3. We will let $K = k(\varepsilon^{1/5})$, which is an abelian quintic extension of k, and is ramified over k only at λ. So the conductor of K/k must be λ to a power that is at least 4. However, $\varepsilon^2 \equiv -1 \bmod \lambda^2$, so it is easy to see that $\varrho = ((\varepsilon^{1/5})^2 + 1)^3/\lambda$ is an integer in K, and a calculation shows that the λ-part of the polynomial discriminant of ϱ over k is λ^{16}. Hence the conductor-discriminant formula shows that K/k has precise conductor λ^4, and the relative discriminant $D_{K/k}$ is λ^{16} (indeed this shows that K is the ray class field of k of conductor λ^4). So the discriminant of K is $D_K = N(\lambda^{16})(5^3)^5 = 5^{31}$. Note that K is normal over $\mathbb{Q}$ (since the conjugate of ε is $-1/\varepsilon$), but is not abelian. Hence $K \neq k(\zeta^{1/5})$, so K has $W_K = 10$ roots of unity.

To compute the class number h_K of K, we let H denote the Hilbert class field of K, which has degree $20h_K$ over $\mathbb{Q}$. We have $D_H^{1/[H:\mathbb{Q}]} = D_K^{1/[K:\mathbb{Q}]} = 5^{31/20} \approx 12.11723433...$. Therefore the Odlyzko bound [W] gives us $20h_K < 60$, so $h_K = 1$ or 2. If 2 divides h_K, then since 2 does not divide the class number of any proper subfield of K containing k, and the order of 2 mod $5 = [K:k]$ is 4, by [W, Theorem 10.8], the 2-rank of the ideal class group of K is a multiple of 4. Hence $h_K = 1$.

3. THE UNITS OF K

Let C be the curve of genus 2 defined by $y^2 = x^5 + 1/4$ over $\mathbb{Q}$, and J its Jacobian. Let ∞ denote the lone point at infinity on C. There is an automorphism $[\zeta]: C \to C$ defined by $[\zeta](x, y) = (\zeta x, y)$ which extends to divisor classes, and gives us an embedding $\iota: \mathcal{O}_k \hookrightarrow \operatorname{End}(J)$. We denote $\iota(\alpha)$ by $[\alpha]$, and let $J[\alpha]$ denote its kernel.

In [Gra3] it is shown that $J[\lambda^3]$ is rational over k, and that $K = k(J[\lambda^4])$ (this is a special case of [Gre]). Indeed, the divisor class P of $(0, 1/2) - \infty$ is in $J[\lambda]$, and if Q is the divisor class of $(1, \sqrt{5}/2) - \infty$, then $[\lambda^2]Q = P$.

Let $R \in J[\lambda^4]$ be chosen so that $[\lambda]R = Q$. Then we can write $G = \operatorname{Gal}(K/k) = \langle \sigma \rangle$, where $\sigma(R) = R + P$. It follows from Lemma 10 of [Gra3] that $\sigma(\varepsilon^{1/5}) = \zeta^2 \varepsilon^{1/5}$.

In the appendix we describe a function $v \in k(J)$ such that $v(R)$ is a root of

$$\mathscr{X}^5 + 5\mathscr{X}^4 + 5\mathscr{X}^2 + 1, \tag{2}$$

so is a unit in K, and a function $\xi \in k(J)$, such that $\xi(R)$ is a root of

$$\begin{aligned} &\mathscr{X}^5 + (1 + 2\zeta - 2\zeta^2 - \zeta^3)\mathscr{X}^4 + (3 + 11\zeta + 14\zeta^2 + 2\zeta^3)\mathscr{X}^3 \\ &\quad + (-6 - 7\zeta + 7\zeta^2 + 11\zeta^3)\mathscr{X}^2 \\ &\quad + (-8 - 16\zeta - 14\zeta^2 - 2\zeta^3)\mathscr{X} + \varepsilon^3, \end{aligned} \tag{3}$$

so is also a unit in K. Along with $\varepsilon^{1/5}$, the conjugates of $\xi(R)$ and $v(R)$ give us 11 units in K with only 2 relations coming from the norm to k of $\xi(R)$ and $v(R)$.

By way of notation, let Σ be any set of absolute values of K containing the archimedean ones. Suppose $\{\varepsilon_1, ..., \varepsilon_{t-1}\}$ is any set of Σ-units, where $t = \#(\Sigma)$. Then we let $R_\Sigma(\varepsilon_1, ..., \varepsilon_{t-1})$ denote $|\mathrm{Det}\,([\,|\varepsilon_i|_{\gamma_j}]_{i,j})|$, where γ_j runs over all but one place of Σ. We also let $R_{\Sigma,K}$ denote the Σ-regulator of K. If Σ is the set of archimedean places of K we omit it from the notation.

A calculation of the roots of (2) and (3), using the action of σ described in the appendix, shows that

$$R(v(R), v(R)^\sigma, v(R)^{\sigma^2}, v(R)^{\sigma^3}, \xi(R), \xi(R)^\sigma, \xi(R)^{\sigma^2}, \xi(R)^{\sigma^3}, \varepsilon^{1/5})$$
$$\approx 7715.338450\ldots .$$

The proof of the following is given in the appendix.

PROPOSITION. $\zeta^{-1} v(R)^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)} = \mu^5$ *for some* $\mu \in K$.

As a corollary we have that

$$R(\mu, v(R), v(R)^\sigma, v(R)^{\sigma^2}, \xi(R), \xi(R)^\sigma, \xi(R)^{\sigma^2}, \xi(R)^{\sigma^3}, \varepsilon^{1/5})$$
$$\approx 1543.067690\ldots, \tag{4}$$

which is an integral multiple of R_K. We will see in the next section that (4) gives R_K, so these 9 units generate the free part of the unit group of K.

4. COMPUTING L-SERIES

Rubin has shown that if his conjecture is true for a given choice of S and T, then his conjecture is still true when S is enlarged. It is easy also to show that Stark's question retains its validity when S is enlarged. Therefore the best testing ground for both is the case where S is minimal. Since k has 2 infinite places, both of which split completely in K, and λ is the only prime of k which ramifies in K, the minimal choice of S is when it consists of the two infinite places and λ, and we will assume henceforth that S is this set.

Let χ_1 be the (imprimitive) trivial character on G. Then

$$L_S(s, \chi_1) = L(s, \chi_1) = \zeta_k(s)(1 - 1/5^s),$$

where ζ_k is the Dedekind ζ-function of k, and

$$\lim_{s\to 0} \frac{L(s,\chi_1)}{s^2} = \log 5 \lim_{s\to 0} \frac{\zeta_k(s)}{s} = \frac{-h_k R_k}{W_k} \log 5 = \frac{-2\,|\log|\varepsilon||\,\log 5}{10}. \tag{5}$$

Now let χ be any non-trivial character on G. For $\rho \in \mathrm{Gal}(k/\mathbb{Q})$, let $\tilde{\rho}$ be any lift of ρ to an element in $\mathrm{Gal}(K/\mathbb{Q})$. Since the map $\mathrm{Gal}(k/\mathbb{Q}) \to \mathrm{Aut}(G)$ given by $\rho \to \{\gamma \to \tilde{\rho}\gamma\tilde{\rho}^{-1}\}$ is surjective (if $\sigma(\varepsilon^{1/5}) = \zeta^2\varepsilon^{1/5}$, and $\rho_2(\zeta) = \zeta^2$, then $\tilde{\rho}_2\sigma\tilde{\rho}_2{}^{-1} = \sigma^3$), we find that no matter which non-trivial χ we pick, it induces the same character on $\mathrm{Gal}(K/\mathbb{Q})$. Therefore $L_S(s,\chi) = L(s,\chi)$ is identical for all non-trivial χ, and χ induces a rational character. (Given the result in [StII], it is not surprising that a tractable example comes from the case of an induced rational character.) In particular,

$$\left(\lim_{s\to 0} \frac{L(s,\chi)}{s^2}\right)^4 = \frac{\lim_{s\to 0} \zeta_K(s)/s^9}{\lim_{s\to 0} \zeta_k(s)/s} = \frac{-h_K R_K/W_K}{-h_k R_k/W_k} = \frac{R_K}{2\,|\log|\varepsilon||}. \tag{6}$$

Since we know an integral multiple of R_K, we really need only to compute a few digits of $\lim_{s\to 0} L(s,\chi)/s^2 = (1/2!)\, L^{(2)}(0,\chi)$ to find out which quotient to take, and then we could easily compute as many digits as we would like of $(1/2!)\, L^{(2)}(0,\chi)$ using our units and the value in (4). If we write $L(s,\chi) = \sum_{n\geqslant 1} a_n/n^s$, which converges for $\mathrm{Re}(s) > 1 - 1/[k:\mathbb{Q}] = 3/4$, then computing the first 10,000 terms at $s=1$ gives that $L(1,\chi)^4$ is approximately 0.64, so R_K is roughly $0.64(2\,|\log|\varepsilon||\,5^{14})/(2\pi)^8$, or about 1548. This is close to the value in (4), so we could assume that (4) gives the regulator. Of course, without a careful error analysis, we would like to have more precision on the computed value of $\sum_{n\geqslant 1} a_n/n$, which converges very slowly. There is a better way to compute $(1/2!)\, L^{(2)}(0,\chi)$, from which we can get 10 significant digits of accuracy by using only the first couple of thousand values of a_n (most of which are zero). The following approach was influenced by [St1].

Let $f(\chi) = \lambda^4$ be the conductor of χ, and set $A = D_k N(f(\chi)) = 5^7$. Since there are two infinite places of k, both complex, it is well-known (see [M]) that

$$\Lambda(s,\chi) = \pi^{-2s-1}\Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{s+1}{2}\right)^2 A^{s/2} L(s,\chi)$$

satisfies a functional equation

$$\Lambda(s,\chi) = w(\chi)\, \Lambda(1-s,\bar{\chi}),$$

where $\bar{\chi}$ is the conjugate of χ. But in our case, $\Lambda(s,\bar{\chi}) = \Lambda(s,\chi)$, so we have

$$\Lambda(s,\chi) = w(\chi)\, \Lambda(1-s,\chi). \tag{7}$$

Applying (7) twice after mapping $s \to 1-s$, we see that the root number $w(\chi)$ is ± 1. The root number is a product of local root numbers, which since k is totally complex and $K/\mathbb{Q}$ is ramified only at 5, are all trivial except for the λ-adic one. But since $f(\chi) = \lambda^4$ is a perfect square, the different of $k/\mathbb{Q}$ is λ^3, and χ is of order dividing 5, the proof on p. 97 of [Tat2] shows that the λ-adic local root number is a fifth root of unity. Hence $w(\chi) = 1$, and

$$\Lambda(s, \chi) = \Lambda(1-s, \chi).$$

Via the duplication formula,

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) = 2^{1-s} \sqrt{\pi}\, \Gamma(s),$$

we have $\Lambda(s, \chi) = 4B^s \Gamma(s)^2 L(s, \chi)$, where $B = 5^{7/2}/(2\pi)^2$.

Starting with the Mellin transform of $K_0(2\sqrt{\alpha t})$, where α is a constant and K_0 is the K_0-Bessel function, we have, for $\mathrm{Re}(s) > 0$,

$$\frac{1}{2} \alpha^{-s} \Gamma(s)^2 = \int_0^\infty K_0(2\sqrt{\alpha t})\, t^s \frac{dt}{t}.$$

So for $\mathrm{Re}(s) > 1$, $\sum_{n \geqslant 1} a_n/n^s$ converges absolutely, and we have

$$\frac{1}{8} \Lambda(s, \chi) = \int_0^\infty \Phi(t)\, t^s \frac{dt}{t},$$

where $\Phi(t) = \sum_{n \geqslant 1} a_n K_0(2\sqrt{nt/B})$. The convergence of Φ and the justification for the exchange of summation and integration come from the fact that for real Y, $K_0(Y)$ grows like $e^{-Y}/Y^{1/2}$ as Y approaches infinity. Now, à la Hecke (using essentially the same proof as [H, pp. 21–23] except with $\Gamma(s)$ replaced by $\Gamma(s)^2$), corresponding to the functional equation for $\Lambda(s, \chi)$ is the functional equation for Φ,

$$\Phi(1/t) = t\Phi(t).$$

So now for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\frac{1}{8} \Lambda(s, \chi) &= \int_0^1 \Phi(t)\, t^s \frac{dt}{t} + \int_1^\infty \Phi(t)\, t^s \frac{dt}{t} \\
&= \int_1^\infty \Phi(1/t)\, t^{-s} \frac{dt}{t} + \int_1^\infty \Phi(t)\, t^s \frac{dt}{t} \\
&= \int_1^\infty \Phi(t)\, t^{-s}\, dt + \int_1^\infty \Phi(t)\, t^s \frac{dt}{t}.
\end{aligned}$$

This last expression holds for all s in the complex plane. Therefore

$$\frac{1}{8}\Lambda(0,\chi)=\int_1^\infty \Phi(t)\left(1+\frac{1}{t}\right)dt=\sum_{n\geqslant 1} a_n\int_1^\infty K_0\left(2\sqrt{\frac{nt}{B}}\right)\left(1+\frac{1}{t}\right)dt.$$

Via numerical integration on *Mathematica*, dropping off all terms for $n>1876$ (the point at which the integral becomes less than 10^{-16}), we get

$$\frac{1}{2!}L^{(2)}(0,\chi)=\frac{1}{4}\Lambda(0,\chi)\approx 6.3278281008... .$$

So we get that

$$R_K=2\,|\log(|\varepsilon|)|\left(\frac{1}{2!}L^{(2)}(0,\chi)\right)^4\approx 1543.067690...,$$

which is an integral divisor of (4) and agrees with (4) to 10 significant digits. Hence R_K has the value in (4) and the units we found do indeed generate the free part of the group of units.

As for computing the coefficients of the Dirichlet series, it is easy to describe the Euler factors of $L(s,\chi)$. If $p\equiv 2, 3 \bmod 5$, then p stays prime in k, and then splits completely in K because p, being real, has an associate congruent to 1 mod λ^4. So the corresponding Euler factor is $(1-1/p^{4s})^{-1}$. If $p\equiv -1 \bmod 5$, then p splits in $\mathbb{Q}(\sqrt{5})$, each factor remains prime in k, and then again, being real, splits completely in K. So the corresponding Euler factor is $(1-1/p^{2s})^{-2}$. Finally, if $p\equiv 1 \bmod 5$, then p splits completely in k, and either each factor splits completely in K contributing an Euler factor of $(1-1/p^s)^{-4}$, or each factor stays prime in K, contributing an Euler factor of $(1+1/p^s+1/p^{2s}+1/p^{3s}+1/p^{4s})^{-1}$. Whether p splits completely in K depends on whether p splits completely in the splitting field of $g(\mathscr{X})=\mathscr{X}^5+5\mathscr{X}^4+5\mathscr{X}^2+1$, which by (2) has $v(R)$ as a root. Since the polynomial discriminant of g is 5^7, $p\neq 5$ splits in the splitting field of g if and only if g factors completely over $\mathbb{Z}/p\mathbb{Z}$.

5. THE KEY EQUALITY

We decided in the last section that our choice for S will be the set containing λ and the two infinite places of k. To specify these, let I be a choice of square root of -1. Define $\mathfrak{p}_1$ to be the place of k which maps ζ to $e^{2\pi I/5}$, and let $\mathfrak{p}_2$ be the place that maps ζ to $e^{4\pi I/5}$.

We now have a choice of which $\mathfrak{P}_i$ to take lying over $\mathfrak{p}_i$, but for verifying the conjectures, it does not really matter. Take $u_i\in K$, $\mathfrak{P}_i$ a place of K lying over $\mathfrak{p}_i$, and $\tau_i\in G$ for $i=1, 2$. Then

$$
\begin{aligned}
R_{(\tau_1(\mathfrak{P}_1),\, \tau_2(\mathfrak{P}_2))}&(u_1 \wedge u_2)\\
&= \operatorname{Det}\left(\left[\sum_{\gamma\in G} \log |u_i^{\gamma}|_{\tau_j(\mathfrak{P}_j)}\, \gamma^{-1}\right]_{i,j}\right)\\
&= \operatorname{Det}\left(\left[\sum_{\gamma\in G} \log |u_i^{\gamma\tau_j^{-1}}|_{\mathfrak{P}_j}\, \gamma^{-1}\right]_{i,j}\right)\\
&= \operatorname{Det}\left(\left[\tau_j^{-1} \sum_{\gamma\in G} \log |u_i^{\gamma\tau_j^{-1}}|_{\mathfrak{P}_j}\, (\gamma\tau_j^{-1})^{-1}\right]_{i,j}\right)\\
&= (\tau_1\tau_2)^{-1}\, R_{(\mathfrak{P}_1,\, \mathfrak{P}_2)}(u_1 \wedge u_2) = R_{(\mathfrak{P}_1,\, \mathfrak{P}_2)}((u_1 \wedge u_2)^{(\tau_1\tau_2)^{-1}}),
\end{aligned}
$$

so the veracity of Stark's question or Rubin's conjecture is independent of the choice of $\mathfrak{P}_i$. We will now make a choice which is convenient for our example.

Recall that $v(R)$ is a root of

$$
g(\mathscr{X}) = \mathscr{X}^5 + 5\mathscr{X}^4 + 5\mathscr{X}^2 + 1,
$$

which has one real root $\mathscr{X} \approx -5.187205764...$. So we take $\mathfrak{P}_1$ and $\mathfrak{P}_2$ to be the places of K lying over $\mathfrak{p}_1$ and $\mathfrak{p}_2$ and lying over the real place of $\mathbb{Q}(v(R))$, and we set $\eta = (\mathfrak{P}_1, \mathfrak{P}_2)$.

We now take $\varepsilon^{1/5}$ to be the fifth-root of ε that is real under $\mathfrak{P}_1$ and $\mathfrak{P}_2$.

Since $\mathrm{Gal}(K/\mathbb{Q})$ is not abelian and has a normal subgroup of order 5, it must have 5 2-Sylow subgroups, and hence there are 5 quintic extensions of $\mathbb{Q}$ in K. These must be $\mathbb{Q}(v(R))$ and its conjugate fields. Since $\varepsilon^{1/5}$ is real under $\mathfrak{P}_1$ and $\mathfrak{P}_2$, so is $\phi = \varepsilon^{1/5} + \varepsilon^{-1/5}$. Since the conjugates of ε are just ε and $-1/\varepsilon$, it follows that ϕ^2 has precisely 5 conjugates, so $\mathbb{Q}(\phi^2)$ is a quintic extension of $\mathbb{Q}$ in K which is real under $\mathfrak{P}_1$ and $\mathfrak{P}_2$, so is $\mathbb{Q}(v(R))$.

For $u_1, u_2 \in K^*$, we define $R_\chi(u_1 \wedge u_2) \in \mathbb{C}$ by $e_\chi R_\eta(u_1 \wedge u_2) = R_\chi(u_1 \wedge u_2) e_\chi$, so

$$
R_\chi(u_1 \wedge u_2) = \operatorname{Det}\left[\sum_{\gamma\in G} \chi(\gamma) \begin{pmatrix} \log |u_1^{\gamma}|_{\mathfrak{P}_1} & \log |u_1^{\gamma}|_{\mathfrak{P}_2} \\ \log |u_2^{\gamma}|_{\mathfrak{P}_1} & \log |u_2^{\gamma}|_{\mathfrak{P}_2} \end{pmatrix}\right].
$$

We extend the definition to $\bigwedge^2 K^*$ by linearity.

We know that $L^{(2)}(0, \chi)$ has the same value for every non-trivial χ, so it will be useful for us to have criteria under which $R_\chi(u_1 \wedge u_2)$ has the same value for every non-trivial χ. Let $\tau \in \mathrm{Gal}(K/\mathbb{Q}(v(R))$ be such that $\tau(\zeta) = \zeta^2$. Note that τ interchanges $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Also since τ generates $\mathrm{Gal}(K/\mathbb{Q}(\phi^2))$, we must have that $\tau(\phi) = -\phi$, so $\tau(\varepsilon^{1/5}) = -\varepsilon^{-1/5}$.

Lemma 1. *Let $u_1, u_2 \in K^*$. Then $R_\chi(u_1 \wedge u_2)$ will have the same value for all $\chi \neq \chi_1$ if*

(a) $u_2^{\sigma-1}$ *is a root of unity, or*

(b) $u_2^{\tau-1}$ *is a root of unity, and* $u_1^{1+\tau}$ *is a root of unity.*

Proof. For part (a), note that if $u_2^{\sigma-1}$ is a root of unity, then $|u_2^{\gamma}|_{\mathfrak{P}_i}$ is the same for all $\gamma \in G$, so for $\chi \neq \chi_1$, $\sum_{\gamma \in G} \chi(\gamma)\,|u_2^{\gamma}|_{\mathfrak{P}_i} = 0$ and $R_\chi(u_1 \wedge u_2)$ vanishes.

For part (b), recall that $\sigma(\varepsilon^{1/5}) = \zeta^2 \varepsilon^{1/5}$. We then note that $\tau\sigma\tau^{-1}(\varepsilon^{1/5}) = \tau\sigma(-\varepsilon^{-1/5}) = \tau(-\zeta^3\varepsilon^{-1/5}) = \zeta^6\varepsilon^{1/5} = \sigma^3(\varepsilon^{1/5})$, so $\tau\sigma\tau^{-1} = \sigma^3$. Hence

$$\begin{aligned}
R_{\chi^2}(u_1 \wedge u_2) &= \operatorname{Det}\left[\sum_{\gamma \in G} \chi^2(\gamma) \begin{pmatrix} \log |u_1^{\gamma}|_{\mathfrak{P}_1} & \log |u_1^{\gamma}|_{\mathfrak{P}_2} \\ \log |u_2^{\gamma}|_{\mathfrak{P}_1} & \log |u_2^{\gamma}|_{\mathfrak{P}_2} \end{pmatrix}\right] \\
&= \operatorname{Det}\left[\sum_{\gamma \in G} \chi(\gamma^2) \begin{pmatrix} \log |u_1^{\gamma}|_{\mathfrak{P}_1} & \log |u_1^{\gamma}|_{\mathfrak{P}_2} \\ \log |u_2^{\gamma}|_{\mathfrak{P}_1} & \log |u_2^{\gamma}|_{\mathfrak{P}_2} \end{pmatrix}\right] \\
&= \operatorname{Det}\left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} \log |u_1^{\gamma^3}|_{\mathfrak{P}_1} & \log |u_1^{\gamma^3}|_{\mathfrak{P}_2} \\ \log |u_2^{\gamma^3}|_{\mathfrak{P}_1} & \log |u_2^{\gamma^3}|_{\mathfrak{P}_2} \end{pmatrix}\right] \\
&= \operatorname{Det}\left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} \log |u_1^{\tau\gamma\tau^{-1}}|_{\mathfrak{P}_1} & \log |u_1^{\tau\gamma\tau^{-1}}|_{\mathfrak{P}_2} \\ \log |u_2^{\tau\gamma\tau^{-1}}|_{\mathfrak{P}_1} & \log |u_2^{\tau\gamma\tau^{-1}}|_{\mathfrak{P}_2} \end{pmatrix}\right] \\
&= \operatorname{Det}\left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} \log |u_1^{\tau\gamma}|_{\mathfrak{P}_2} & \log |u_1^{\tau\gamma}|_{\mathfrak{P}_1} \\ \log |u_2^{\tau\gamma}|_{\mathfrak{P}_2} & \log |u_2^{\tau\gamma}|_{\mathfrak{P}_1} \end{pmatrix}\right],
\end{aligned}$$

since τ interchanges $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Hence

$$R_{\chi^2}(u_1 \wedge u_2) = \operatorname{Det}\left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} \log |u_1^{-\gamma}|_{\mathfrak{P}_2} & \log |u_1^{-\gamma}|_{\mathfrak{P}_1} \\ \log |u_2^{\gamma}|_{\mathfrak{P}_2} & \log |u_2^{\gamma}|_{\mathfrak{P}_1} \end{pmatrix}\right] = R_\chi(u_1 \wedge u_2),$$

by the hypotheses on u_1 and u_2. Since $\chi \to \chi^2$ is a transitive action on the nontrivial χ, all the $R_\chi(u_1 \wedge u_2)$ have the same value for $\chi \neq \chi_1$.

To apply Lemma 1, we will consider

$$\phi_1 = \frac{(\varepsilon^{1/5} + \varepsilon^{-1/5})(\zeta^2\varepsilon^{1/5} + \zeta^3\varepsilon^{-1/5})(\zeta^3\varepsilon^{1/5} + \zeta^2\varepsilon^{-1/5})}{\lambda} = \phi^{1+\sigma+\sigma^4}/\lambda.$$

Since $\varepsilon^{1/5}$ is a unit in K, $\phi = \varepsilon^{1/5} + \varepsilon^{-1/5}$ is an integer whose norm to k is $\varepsilon + \varepsilon^{-1} = \sqrt{5}$. Hence ϕ_1 is an Σ-unit of K, where Σ consists of the archimedean places and the lone prime in K above λ. We let

$$\omega = \frac{v(R)^2}{v(R+P)\,v(R+4P)} = v(R)^{2-\sigma-\sigma^4},$$

and $\phi_2 = \varepsilon^{1/5}\omega^2$, which is a unit.

Note that $\phi_1 \wedge \phi_2 = \phi_1 \wedge \varepsilon^{1/5} + \phi^{\sigma-\sigma^2-\sigma^3+\sigma^4} \wedge v(R)^2$. By part (a) of Lemma 1, $R_\chi(\phi_1 \wedge \varepsilon^{1/5}) = 0$ for $\chi \neq \chi_1$. Since τ fixes $v(R)$, we have $v(R)^{\tau-1} = 1$. Also

$$\tau(\phi^{\sigma-\sigma^2-\sigma^3+\sigma^4}) = \tau\left(\frac{(\zeta^2\varepsilon^{1/5}+\zeta^3\varepsilon^{-1/5})(\zeta^3\varepsilon^{1/5}+\zeta^2\varepsilon^{-1/5})}{(\zeta^4\varepsilon^{1/5}+\zeta\varepsilon^{-1/5})(\zeta\varepsilon^{1/5}+\zeta^4\varepsilon^{-1/5})}\right)$$
$$= \frac{(-\zeta^4\varepsilon^{-1/5}-\zeta\varepsilon^{1/5})(-\zeta\varepsilon^{-1/5}-\zeta^4\varepsilon^{1/5})}{(-\zeta^3\varepsilon^{-1/5}-\zeta^2\varepsilon^{1/5})(-\zeta^2\varepsilon^{-1/5}-\zeta^3\varepsilon^{1/5})} = \phi^{-\sigma+\sigma^2+\sigma^3-\sigma^4}.$$

So by part (b) of Lemma 1, $R_\chi(\phi^{\sigma-\sigma^2-\sigma^3+\sigma^4} \wedge v(R)^2)$ is the same for all $\chi \neq \chi_1$, so the same is true of $R_\chi(\phi_1 \wedge \phi_2)$.

We are now in a position to state and attack the key equality, which is something out of the Stark/Sands framework [StI–IV, Sa4]. Given our L-series calculations, it is easy to verify the equation numerically, but we will give a proof of equality.

THEOREM 1. (a) (*Key Equality*) *With k, K, S, $\mathfrak{P}_1$, and $\mathfrak{P}_2$ as above, for all characters χ on G,*

$$\frac{1}{2!} L_S^{(2)}(0,\chi) = \frac{1}{W_K} \operatorname{Det}\left[\sum_{\gamma\in G} \chi(\gamma) \begin{pmatrix} \log|\phi_1^\gamma|_{\mathfrak{P}_1} & \log|\phi_1^\gamma|_{\mathfrak{P}_2} \\ \log|\phi_2^\gamma|_{\mathfrak{P}_1} & \log|\phi_2^\gamma|_{\mathfrak{P}_2} \end{pmatrix}\right].$$

Equivalently,

$$\theta_{S,\varnothing}^{(2)}(0) = \frac{1}{W_K} R_\eta(\phi_1 \wedge \phi_2).$$

(b) *$K(\phi_2^{1/W_K})$ is abelian over k.*

For the proof we need a lemma which is a generalization of the classical Dedekind determinant [W].

LEMMA 2. *Let Υ be any finite abelian group of order n. For some m, let φ_{ij}, $1 \leqslant i, j \leqslant m$, be a set of complex-valued functions on Υ. For $1 \leqslant i, j \leqslant m$, let $M_{ij} = [\varphi_{ij}(a^{-1}b)]_{a,b\in\Upsilon}$, and let M be the $mn\times mn$ matrix, consisting of m^2 blocks of size $n\times n$, whose ij*th *block is M_{ij}. Then*

$$\operatorname{Det}(M) = \prod_{\chi\in\hat{\Upsilon}} \operatorname{Det}\left(\sum_{a\in\Upsilon} \chi(a)[\varphi_{ij}(a)]_{1\leqslant i,j\leqslant m}\right).$$

Proof. Let Ω be the n-dimensional space of complex-valued functions on Υ. One basis for Ω consists of the characters $\hat{\Upsilon}$, and another the characteristic functions $\{\delta_b \mid b \in \Upsilon\}$, where $\delta_b(a) = 1$ if $a = b$, and is zero otherwise.

Let $\psi_i : \Omega \to \Omega^m$ be the embedding into $0^{i-1} \times \Omega \times 0^{m-i}$. We let $\Omega_{(i)} = \psi_i(\Omega)$, and for any $\varphi \in \Omega$, let $\varphi_{(i)} = \psi_i(\varphi)$. To define a linear transformation on Ω^m it suffices to define it on each $\Omega_{(i)}$. First, for $a \in \Upsilon$, let $\mathscr{T}_a$ be the linear transformation on Ω defined by $(\mathscr{T}_a \varphi)(b) = \varphi(ab)$. Then we define a linear transformation $\mathscr{T}$ on Ω^m by defining for each $\varphi \in \Omega$ and each i:

$$\mathscr{T}(\varphi_{(i)}) = \sum_{a \in \Upsilon} \sum_{j=1}^{m} \varphi_{ij}(a)(\mathscr{T}_a \varphi)_{(j)}.$$

Then with respect to the basis $(\delta_b)_{(i)}$ we have

$$\mathscr{T}((\delta_b)_{(i)}) = \sum_{a \in \Upsilon} \sum_{j=1}^{m} \varphi_{ij}(a)(\delta_{a^{-1}b})_{(j)},$$

so the determinant of $\mathscr{T}$ is the determinant of M. On the other hand, if we let $\Omega_\chi = \bigoplus_i \mathbb{C}\chi_{(i)}$, then $\Omega^m = \bigoplus_\chi \Omega_\chi$, and

$$\mathscr{T}(\chi_{(i)}) = \sum_{a \in \Upsilon} \sum_{j=1}^{m} \varphi_{ij}(a)\, \chi(a)\, \chi_{(j)},$$

so Ω_χ is an eigenspace for $\mathscr{T}$. So the determinant of $\mathscr{T}$ is the product of the determinant of the induced transformation on each Ω_χ, and that determinant is

$$\mathrm{Det}\left(\sum_{a \in \Upsilon} \chi(a)[\varphi_{ij}(a)]_{1 \leqslant i,\, j \leqslant m} \right).$$

Proof of Theorem 1. By Lemma 2, with $m = 2$, $\varphi_{ij}(\gamma) = \log |\phi_i|_{\gamma \mathfrak{P}_j}$, we know that

$$R_{\Sigma, K}(\phi_1, \phi_1^\sigma, \phi_1^{\sigma^2}, \phi_1^{\sigma^3}, \phi_1^{\sigma^4}, \phi_2, \phi_2^\sigma, \phi_2^{\sigma^2}, \phi_2^{\sigma^3}, \phi_2^{\sigma^4}) = \left| \prod_{\chi \in \hat{G}} R_\chi(\phi_1 \wedge \phi_2) \right|,$$

and is an integral multiple of $R_{\Sigma, K} = (\log 5)\, R_K$. A calculation shows that this multiple is 10^4, so $|\prod_{\chi \in \hat{G}} R_\chi(\phi_1 \wedge \phi_2)| = 10^4 (\log 5)\, R_K$.

Note that

$$\begin{aligned} R_{\chi_1}(\phi_1 \wedge \phi_2) &= \mathrm{Det} \begin{bmatrix} \log |N_{K/k}(\phi_1)_{\mathfrak{p}_1} & \log |N_{K/k}(\phi_1)_{\mathfrak{p}_2} \\ \log |N_{K/k}(\phi_2)_{\mathfrak{p}_1} & \log |N_{K/k}(\phi_2)_{\mathfrak{p}_2} \end{bmatrix} \\ &= \mathrm{Det} \begin{bmatrix} \log |(\varepsilon + \varepsilon^{-1})^3/\lambda^5|_{\mathfrak{p}_1} & \log |(\varepsilon + \varepsilon^{-1})^3/\lambda^5|_{\mathfrak{p}_2} \\ \log |\varepsilon|_{\mathfrak{p}_1} & \log |\varepsilon|_{\mathfrak{p}_2} \end{bmatrix}. \end{aligned}$$

Now $|\varepsilon|_{\mathfrak{p}_1} = |\varepsilon|^2$, and $|\varepsilon|_{\mathfrak{p}_2} = |-1/\varepsilon|^2$. Also $\varepsilon + \varepsilon^{-1} = \sqrt{5}$, so we have that

$$R_{\chi_1}(\phi_1 \wedge \phi_2) = -2\, |\log |\varepsilon||\, (6 \log 5 - 5 \log(|\lambda|_{\mathfrak{p}_1} |\lambda|_{\mathfrak{p}_2})|.$$

But $|\lambda|_{\mathfrak{p}_1} |\lambda|_{\mathfrak{p}_2} = (1-\zeta)(1-\zeta^4)(1-\zeta^2)(1-\zeta^3) = 5$, so

$$R_{\chi_1}(\phi_1 \wedge \phi_2) = -2\,|\log|\varepsilon||\log 5,$$

and by (5) the key equality is established for $\chi = \chi_1$. Hence by (5) and (6) we have

$$\left|\prod_{\chi \neq \chi_1} R_\chi(\phi_1 \wedge \phi_2)\right| = \frac{10^4 R_K}{2\,|\log|\varepsilon||} = 10^4 \left(\frac{1}{2!} L^{(2)}(0, \chi)\right)^4,$$

for any $\chi \neq \chi_1$. But for any $\chi \neq \chi_1$, $R_\chi(\phi_1 \wedge \phi_2)$ has the same value, so for $\chi \neq \chi_1$

$$\frac{(1/10)\, R_\chi(\phi_1 \wedge \phi_2)}{(1/2!)\, L^{(2)}(0, \chi)} = I^m,$$

for some $m \in \mathbb{Z}$. A calculation shows that this ratio is 1, so for all $\chi \neq \chi_1$,

$$\frac{1}{10} R_\chi(\phi_1 \wedge \phi_2) = \frac{1}{2!} L^{(2)}(0, \chi),$$

as desired, which gives us part (a)

For part (b) we need only show that $K(\phi_2^{1/2})/k$ and $K(\phi_2^{1/5})/k$ are abelian. But the former is easy since $K(\phi_2^{1/2})$ is $k(\varepsilon^{1/10})$, which is a Kummer extension of k. So we need only show that $K(\phi_2^{1/5})/k$ is abelian.

It suffices to prove normality since all groups of order 5 or 25 are abelian. Recall that $\sigma(\varepsilon^{1/5}) = \zeta^2 \varepsilon^{1/5}$. It suffices to show that $\sigma(\phi_2)/\phi_2 \in (K^*)^5$. Since $\sigma(\varepsilon^{1/5})/\varepsilon^{1/5} = \zeta^2$, we just have to show that $\zeta\omega^{\sigma-1}$ is a fifth power. But by the proposition, we know that $\zeta^{-1} v(R)^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)} = \mu^5$, and $\omega = v(R)^{2-\sigma-\sigma^4}$, so $\zeta\omega^{\sigma-1} = \mu^{5(2\sigma+\sigma^3+\sigma^4)}$, since $(2-\sigma-\sigma^4)(\sigma-1) = (1-\sigma^2)(1-\sigma^3)(1-\sigma^4)(2\sigma+\sigma^3+\sigma^4)$.

6. VERIFICATION OF THE CONJECTURES FROM THE KEY EQUALITY

We will now verify Stark's question and Rubin's conjecture for our choice of k, K, S, and all appropriate T.

Given Theorem 1, to affirm Stark's question we now just need to set $\varepsilon_1 = \phi_1^{10a}\phi_2^b$, and $\varepsilon_2 = \phi_1^{10c}\phi_2^d$, with $ad - bc = 1$. To affirm the "follow-up question," a choice like $a = 1$, $b = 3$, $c = 2$, $d = 7$ will give $K(\varepsilon_1^{1/W_K}) = K(\varepsilon_2^{1/W_K})$, and then for $i = 1, 2$, $\varepsilon_i^\sigma \mathcal{O}_K = \varepsilon_i \mathcal{O}_K$ for all $\sigma \in G$, since $\phi_1^{\sigma-1}$ and ϕ_2 are units.

To verify Rubin's conjecture we need to do a little more work. Note that since $\lambda \in S$, $U_{S,T}$ will be torsion-free if T is any finite, non-empty set of primes disjoint from S, except the set $\{2\}$.

THEOREM 2. *Let T be any set disjoint from S, such that $U_{S,T}$ is torsion-free. Then Rubin's conjecture holds for our choice of k, K, S, and T.*

Proof. With our choice of k, K, and S, for every character χ on G, $r(\chi)=2=r$, so $\Lambda_{S,T}=\bigwedge_0^2 U_{S,T}$. Let $V_{S,T}=U_{S,T}\cap k$. Let $\mathbf{N}=1+\sigma+\sigma^2+\sigma^3+\sigma^4$ be the norm element in $\mathbb{Z}[G]$. For any $u\in V_{S,T}$, $u^{1-\sigma}=1$, so if $\phi\in \mathrm{Hom}(U_{S,T}, \mathbb{Z}[G])$, then $(1-\sigma)\,\phi(u)=0$, so $\phi(u)\in\mathbf{N}\mathbb{Z}[G]$. Since $\mathbf{N}^2=5\mathbf{N}$, we have for any $u_1, u_2\in V_{S,T}$ that $(\phi_1\wedge\phi_2)(u_1\wedge u_2)\in \mathbf{N}^2\mathbb{Z}[G]=5\mathbf{N}\mathbb{Z}[G]$, so $\frac{1}{5}u_1\wedge u_2\in\bigwedge_0^2 U_{S,T}$. So if we set $W_{S,T}=\frac{1}{5}\bigwedge^2 V_{S,T}+\bigwedge^2 U_{S,T}$, then $W_{S,T}\subseteq\bigwedge_0^2 U_{S,T}$. What we will prove is that there exists an $\varepsilon_{S,T}\in W_{S,T}$ so that Rubin's conjecture holds.

We will proceed by induction on the number of primes in T. We will first show the theorem for $T=\{\mathfrak{q}\}$. Let $\varepsilon_{S,\varnothing}=\frac{1}{10}\phi_1\wedge\phi_2$. We have

$$\begin{aligned}\theta^{(2)}_{S,\{\mathfrak{q}\}}(0)&=(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}\,N\mathfrak{q})\,\theta^{(2)}_{S,\varnothing}(0)\\ &=(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}\,N\mathfrak{q})\,R_\eta(\varepsilon_{S,\varnothing})=R_\eta((1-\mathrm{Fr}_{\mathfrak{q}}^{-1}\,N\mathfrak{q})\,\varepsilon_{S,\varnothing}).\end{aligned}$$

We will use throughout that for any root of unity ρ of K of order r, and any element ξ of K^*, we have

$$\rho\wedge\xi=\frac{1}{r}\,\rho^r\wedge\xi=\frac{1}{r}\,1\wedge\xi=0.$$

We will also use the argument in the proof of [Tat1, IV.1.2] which shows for any S-unit $u\in K$ such that $K(u^{1/10})$ is abelian over k, that for any prime $\mathfrak{p}$ of K which is not in S and whose norm is prime to 10,

$$(1-\mathrm{Fr}_{\mathfrak{p}}^{-1}\,N\mathfrak{p})\,u=\alpha^{10},$$

for some S-unit $\alpha\in K$ with $\alpha\equiv 1 \bmod \mathfrak{p}$.

We first assume that $\mathfrak{q}$ stays prime in K. Then $\mathcal{O}_K/\mathfrak{q}\mathcal{O}_K$ is a field, hence has a cyclic multiplicative group, so there are relatively prime integers a and b such that $\phi_1^a\phi_2^b\equiv 1 \bmod \mathfrak{q}\mathcal{O}_K$. We will show that $(a, 5)=1$. For if $5\mid a$, then $(5, b)=1$, and ϕ_2 would be a fifth power mod $\mathfrak{q}\mathcal{O}_K$, so $N_{K/k}(\phi_2)=\varepsilon$ would be a fifth power mod $\mathfrak{q}$. But then $\mathfrak{q}$ would split completely in K/k. Hence there are integers c and d such that $ad-bc=1$ and $5\mid c$. Say $c=5c'$. So we have

$$\tfrac{1}{10}\phi_1\wedge\phi_2=\tfrac{1}{10}\phi_1^a\phi_2^b\wedge\phi_1^c\phi_2^d=\tfrac{1}{2}\phi_1^a\phi_2^b\wedge\phi_1^{c'}+\tfrac{1}{10}\phi_1^a\phi_2^b\wedge\phi_2^d.$$

Suppose for the moment that c' is even. Then $(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}N\mathfrak{q})\,\phi_1\equiv 1 \bmod \mathfrak{q}\mathcal{O}_K$, and since $K(\phi_2^{1/10})$ is abelian over k, it follows by Tate's argument above that $(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}N\mathfrak{q})\,\phi_2=\alpha^{10}$ for some $\alpha\in U_{S,\{\mathfrak{q}\}}$. Hence if we set

$\varepsilon_{S,\{\mathfrak{q}\}} = (1-\mathrm{Fr}_{\mathfrak{q}}^{-1} N\mathfrak{q})\,\frac{1}{10}\phi_1 \wedge \phi_2$, then $\varepsilon_{S,\{\mathfrak{q}\}} \in \bigwedge^2 U_{S,\{\mathfrak{q}\}} \subseteq W_{S,\{\mathfrak{q}\}}$, and $R_\eta(\varepsilon_{S,\{\mathfrak{q}\}}) = \theta^{(2)}_{S,\{\mathfrak{q}\}}(0)$.

If c' is odd, then we still want to show that $(1-\mathrm{Fr}_{\mathfrak{q}}^{-1} N\mathfrak{q})\,\frac{1}{2}\phi_1 \wedge \phi_2 \in \bigwedge^2 U_{S,\{\mathfrak{q}\}}$. Since $\phi_2 = \varepsilon(\omega/\varepsilon^{2/5})^2$, it suffices by an argument similar to the one above to show that $(1-\mathrm{Fr}_{\mathfrak{q}}^{-1} N\mathfrak{q})\,\frac{1}{2}\phi_1 \wedge \varepsilon \in \bigwedge^2 U_{S,\{\mathfrak{q}\}}$. But

$$\tfrac{1}{2}\phi_1 \wedge \varepsilon = \tfrac{1}{2}\phi_1 \wedge (\varepsilon^{1/5})^{\mathbf{N}} = \tfrac{1}{2}\phi_1^{\mathbf{N}} \wedge \varepsilon^{1/5} = \tfrac{1}{10}\phi_1^{\mathbf{N}} \wedge \varepsilon. \tag{8}$$

So there are integers a, b, c, d with $ad-bc=1$ such that $(\phi_1^{\mathbf{N}})^a\,\varepsilon^b \equiv 1 \bmod \mathfrak{q}$, and

$$\tfrac{1}{10}\phi_1^{\mathbf{N}} \wedge \varepsilon = \tfrac{1}{10}(\phi_1^{\mathbf{N}})^a\,\varepsilon^b \wedge (\phi_1^{\mathbf{N}})^c\,\varepsilon^d.$$

But $K(((\phi_1^{\mathbf{N}})^c\,\varepsilon^d)^{1/10})$ is abelian over k, so using Tate's argument as above, we have

$$(1-\mathrm{Fr}_{\mathfrak{q}}^{-1} N\mathfrak{q})\,\tfrac{1}{10}(\phi_1^{\mathbf{N}})^a\,\varepsilon^b \wedge (\phi_1^{\mathbf{N}})^c\,\varepsilon^d \in \textstyle\bigwedge^2 U_{S,\{\mathfrak{q}\}}.$$

If $\mathfrak{q}$ splits in K, we have to work harder. First note that if $\mathfrak{J}$ is the ideal of integers n such that $n(1-\mathrm{Fr}_{\mathfrak{q}}^{-1} N\mathfrak{q})\,\varepsilon_{S,\varnothing} \in W_{S,\{\mathfrak{q}\}}$, then we need to show that $\mathfrak{J}$ is the unit ideal. Equivalently, for every prime p, we need to find an $n = n_p \in \mathfrak{J}$ with n_p prime to p. Since $U_S/U_{S,\{\mathfrak{q}\}}$ is finite, this is the same as saying that

$$(1-N\mathfrak{q})\varepsilon_{S,\varnothing} \in W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_p \subseteq \textstyle\bigwedge_0^2 U_{S,\{\mathfrak{q}\}} \otimes \mathbb{Q}_p \tag{9}$$

for every prime p, where $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are respectively the ring of p-adic integers and the field of p-adic numbers.

When $p \neq 2, 5$, (9) is almost immediate. Consider the composite

$$\psi : U_S \to (O_K/\mathfrak{q})^\times \to (O_K/\mathfrak{q})^\times \otimes \mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}[G],$$

where $p^n \parallel N\mathfrak{q}-1$. Arguing analogously to the case where $\mathfrak{q}$ stays prime, it suffices to find $a, b, c, d \in \mathbb{Z}_p[G]$ such that $\psi(\phi_1^a\phi_2^b)=0$, and $ad-bc=1$, since $(1-\mathrm{Fr}_{\mathfrak{q}}^{-1} N\mathfrak{q})\,\phi_1^c\phi_2^d \in U_{S,\{\mathfrak{q}\}}$. Note that since 5 is invertible in $\mathbb{Z}_p$, we have isomorphisms

$$\begin{aligned}\mathbb{Z}_p[G] &\cong \mathbb{Z}_p[x]/(x^5-1)\\ &\cong \mathbb{Z}_p[x]/(x-1) \times \mathbb{Z}_p[x]/(x^4+x^3+x^2+x+1) \cong \mathbb{Z}_p \times \mathbb{Z}[\zeta] \otimes \mathbb{Z}_p.\end{aligned}$$

So $\mathbb{Z}_p[G]$ is a product of discrete valuation domains; call them R_i. Hence if ψ_i is the composite of ψ and the projection $\mathbb{Z}/p^n\mathbb{Z}[G] \to R_i/p^n R_i$, it suffices to find $a_i, b_i, c_i, d_i \in R_i$, with $a_id_i - b_ic_i = 1$, such that $\psi_i(\phi_1^{a_i}\phi_2^{b_i}) = 0$. Since R_i is a discrete valuation ring, either $\psi_i(\phi_1)$ divides

$\psi_i(\phi_2)$, or vice versa. Without loss of generality, we can assume the latter, and take $a_i = 1$, $b_i = -\psi_i(\phi_1)/\psi_i(\phi_2)$, $c_i = 0$, $d_i = 1$.

When $p = 2$, then 5 is a unit in $\mathbb{Z}_p$, so as earlier, we rewrite

$$\tfrac{1}{2}\phi_1 \wedge \phi_2 = \tfrac{1}{2}\phi_1 \wedge \varepsilon(\omega/\varepsilon^{2/5})^2 = \tfrac{1}{2}\phi_1 \wedge \varepsilon + \phi_1 \wedge (\omega/\varepsilon^{2/5}).$$

The second summand we handle just as in the $p \neq 2, 5$ case, and by (8), the first summand is $\frac{1}{10}\phi_1^{\mathbf{N}} \wedge \varepsilon$. So again, there exists integers a, b, c, d with $ad - bc = 1$ such that $\phi_1^{\mathbf{N}a}\varepsilon^b \equiv 1 \bmod \mathfrak{q}$, and then it suffices to note that $\pm((1 - N\mathfrak{q})/2)\, \phi_1^{\mathbf{N}c}\varepsilon^d \equiv 1 \bmod \mathfrak{q}$.

When $p = 5$, we still have an injection

$$\mathbb{Z}_5[G] \to \mathbb{Z}_5[x]/(x-1) \times \mathbb{Z}_5[x]/(x^4 + x^3 + x^2 + x + 1) \cong \mathbb{Z}_5 \times \mathbb{Z}[\zeta]_\lambda,$$

into a product of discrete valuation rings; let $R_1 = \mathbb{Z}_5$ and $R_2 = \mathbb{Z}[\zeta]_\lambda$. The image is the set of pairs (α, β) where $\alpha \equiv \beta \bmod \lambda$. Let ψ be as above, and ψ_i be the composite of ψ and the projections π: $\mathbb{Z}/5^n\mathbb{Z}[G] \to R_i/5^n R_i$ for $i = 1, 2$. Note that $\psi(\zeta)$ is non-trivial, of order 5, and fixed by G. So it generates the subgroup $5^{n-1}\mathbf{N}\mathbb{Z}/5^n\mathbb{Z}$ in $\mathbb{Z}/5^n\mathbb{Z}[G]$. This subgroup is precisely the kernel of $\mathbb{Z}/5^n\mathbb{Z}[G] \to R_1/5^n R_1 \times R_2/5^n R_2$. Note also that by Tate's argument, $(1 - N\mathfrak{q})\, \phi_2 = \alpha^5$, with $\alpha \in U_{S, \{\mathfrak{q}\}}$. Hence $\phi_2^{5^{n-1}} \zeta^j \in U_{S, \{\mathfrak{q}\}} \otimes \mathbb{Z}_5$, for some j. Therefore $\psi(\phi_2)$ is an integral multiple of $\mathbf{N}$ mod 5.

Note that the units of $\mathbb{Z}/5^n\mathbb{Z}[G]$ are precisely those x with $\pi_1(x) \neq 0$ mod 5 (which occurs precisely when $\pi_2(x) \neq 0 \bmod \lambda$).

We have 2 cases.

Case 1 ($\psi(\phi_1)$ Is a Unit in $\mathbb{Z}/5^n\mathbb{Z}[G]$). Recall that $(\varepsilon^{1/5})^{\sigma - 1} = \zeta^2$, and that $\psi(\zeta) = a\mathbf{N}5^{n-1}$ for some non-zero $a \in \mathbb{Z}/5\mathbb{Z}$. So

$$(\sigma - 1)\, \psi(\varepsilon^{1/5}) = 2a\mathbf{N}5^{n-1} = 2a(\mathbf{N} - 5)\, 5^{n-1},$$

hence

$$\psi(\varepsilon^{1/5}) = 2a(1 - \sigma^2)(1 - \sigma^3)(1 - \sigma^4)\, 5^{n-1} + \mathbf{N}x, \tag{10}$$

for some $x \in \mathbb{Z}/5^n\mathbb{Z}[G]$. We also saw in the proof of Theorem 1 that $\phi_2^{\sigma - 1} = u^5$, where $u = \mu^{2(2\sigma + \sigma^3 + \sigma^4)}$. We need to compute $\mu^{\mathbf{N}}$. Recall from the proposition that $\mu^5 = \zeta^{-1} v(R)^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)}$ and from (2) that $(-v(R))^{\mathbf{N}} = 1$. By Hilbert's Theorem 90, there is a $v \in K$ such that $v^{1-\alpha} = -v(R)$. Then $\mu^5 = \zeta^{-1}v^{5-\mathbf{N}}$, so $v^5/\mu^5 = v^{\mathbf{N}}\zeta \in k$. Hence, by Kummer theory, $v = \mu(\varepsilon^{1/5})^i\, \xi$ for some $\xi \in k$ and some integer i. Hence $v(R) = -\mu^{1-\sigma}\zeta^{-2i}$, and so

$$\mu^5 = \zeta^{-1}(-\mu^{1-\sigma}\zeta^{-2i})^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)} = \zeta^{-1}\mu^{5-\mathbf{N}}.$$

Hence $\mu^{\mathbf{N}} = \zeta^{-1}$, and as a consequence, $u^{\mathbf{N}} = \zeta^2$. Therefore $\mathbf{N}\psi(u) = 2a\mathbf{N}5^{n-1}$, and hence

$$\psi(u) = 2a5^{n-1} + (1-\sigma)\, y, \tag{11}$$

for some $y \in \mathbb{Z}/5^n\mathbb{Z}[G]$. Now choose X, $Y \in \mathbb{Z}_5[G]$ and $A \in \mathbb{Z}$ such that $X \equiv x \mod 5^n$, $Y \equiv y \mod 5^n$, and $A \equiv a \mod 5$. So if we set $\Psi(u) = 2A5^{n-1} + (1-\sigma)\, Y$, and $\Psi(\varepsilon^{1/5}) = 2A(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)\, 5^{n-1} + \mathbf{N}X$, and let $\Psi(\phi_1)$ be any unit in $\mathbb{Z}_5[G]$ such that $\Psi(\phi_1) \equiv \psi(\phi_1) \mod 5^n$, we get

$$\mathbf{N}\Psi(u) - (\sigma - 1)\, \Psi(\varepsilon^{1/5}) = 2A5^n.$$

We want to show $5^{n-1}\phi_1 \wedge \phi_2 \in W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5$, or equivalently, since $2A$ and $\Psi(\phi_1)$ are units in $\mathbb{Z}_5[G]$, that

$$\tfrac{1}{5}\phi_1 \wedge \phi_2^{\mathbf{N}\Psi(u)/\Psi(\phi_1) - (\sigma-1)\,\Psi(\varepsilon^{1/5})/\Psi(\phi_1)} \tag{12}$$

is in $W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5$. But we can rewrite (12) as

$$\begin{aligned}
&\tfrac{1}{5}\phi_1 \wedge \varepsilon^{\Psi(u)/\Psi(\phi_1)} + \tfrac{1}{5}\phi_1 \wedge (u^5)^{-\Psi(\varepsilon^{1/5})/\Psi(\phi_1)} \\
&\quad = \phi_1 \wedge (\varepsilon^{1/5})^{\Psi(u)/\Psi(\phi_1)} + \phi_1 \wedge u^{-\Psi(\varepsilon^{1/5})/\Psi(\phi_1)} \\
&\quad = \phi_1^{\Psi(u)/\Psi(\phi_1)} \wedge \varepsilon^{1/5} + \phi_1^{-\Psi(\varepsilon^{1/5})/\Psi(\phi_1)} \wedge u \\
&\quad = \phi_1^{\Psi(u)/\Psi(\phi_1)} u^{-1} \wedge \varepsilon^{1/5}\phi_1^{-\Psi(\varepsilon^{1/5})/\Psi(\phi_1)},
\end{aligned}$$

since clearly $\phi_1^{\Psi(u)/\Psi(\phi_1)} \wedge \phi_1^{-\Psi(\varepsilon^{1/5})/\Psi(\phi_1)} = 0$, and $u^{-1} \wedge \varepsilon^{1/5} = -\tfrac{1}{5}u^5 \wedge \varepsilon^{1/5} = \tfrac{1}{5}\phi_2^{1-\sigma} \wedge \varepsilon^{1/5} = \tfrac{1}{5}\phi_2 \wedge (\varepsilon^{1/5})^{1-\sigma} = \tfrac{1}{5}\phi_2 \wedge \zeta^{-2} = 0$ as well. Finally, by (10) and (11), one sees that $\psi(\phi_1^{\Psi(u)/\Psi(\phi_1)} u^{-1}) = \psi(\varepsilon^{1/5}\phi_1^{-\Psi(\varepsilon^{1/5})/\Psi(\phi_1)}) = 0$, as desired.

Case 2 ($\psi_1(\phi_1) \equiv 0 \mod 5$ and $\psi_2(\phi_1) \equiv 0 \mod \lambda$). If $n = 1$, (11) shows that $\psi(u)$ is a unit. Since $u \wedge \phi_2 = \tfrac{1}{5}u^5 \wedge \phi_2 = \tfrac{1}{5}\phi_2^{\sigma-1} \wedge \phi_2 = 0$, we can apply the argument of Case 1 with ϕ_1 replaced by $\phi_1 u$, since then $\psi(\phi_1 u)$ is a unit.

So without loss of generality, we can assume $n \geqslant 2$, in which case, by (11), $\psi_1(u) \equiv 0 \mod 5$ and $\psi_2(u) \equiv 0 \mod \lambda$.

We want to show $5^{n-1}\phi_1 \wedge \phi_2 \in W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5$. We rewrite this as

$$5^{n-2}\phi_1 \wedge \phi_2^{\mathbf{N}+5-\mathbf{N}} = 5^{n-2}\phi_1 \wedge \varepsilon - 5^{n-2}\mathbf{N}'\phi_1 \wedge \phi_2^{\sigma-1},$$

where $\mathbf{N}' = (1-\sigma^2)(1-\sigma^3)(1-\sigma^4)$, so $\mathbf{N}'(1-\sigma) = 5 - \mathbf{N}$. Recall that $\phi_2^{\sigma-1} = u^5$ where u is an S-unit in K, and $\varepsilon = (\varepsilon^{1/5})^{\mathbf{N}}$, so

$$5^{n-1}\phi_1 \wedge \phi_2 = 5^{n-3}\phi_1^{\mathbf{N}} \wedge \varepsilon - 5^{n-1}\mathbf{N}'\phi_1 \wedge u. \tag{13}$$

Consider the first summand in (13). Since by assumption $\psi_1(\phi_1) \equiv 0$ mod 5, $\phi_1^{\mathbf{N}}$ is a fifth power mod $\mathfrak{q}$, as is ε, since $\mathfrak{q}$ splits in K. So there exists a, b, c, d in $\mathbb{Z}$, with $ad - bc = 5^{n-2}$, such that $\psi_1(\phi_1^{\mathbf{N}a}\varepsilon^b) \equiv \psi_1(\phi_1^{\mathbf{N}c}\varepsilon^d) \equiv 0$ mod 5^{n-1}. Since ψ_2 of a norm vanishes, we also have $\psi(\phi_1^{\mathbf{N}a}\varepsilon^b) \equiv \psi(\phi_1^{\mathbf{N}c}\varepsilon^d) \equiv 0$ mod 5^{n-1}. Since $\psi(\phi_1^{\mathbf{N}})$ and $\psi(\varepsilon)$ also are multiples of $\mathbf{N}$, $\psi(\phi_1^{\mathbf{N}})$ and $\psi(\varepsilon)$ are in the subgroup generated by $\mathbf{N}5^{n-1}$. Hence there exists integers i and j such that $\psi(\phi_1^{\mathbf{N}a}\varepsilon^b\zeta^i) = \psi(\phi_1^{\mathbf{N}c}\varepsilon^d\zeta^j) = 0$, and so

$$5^{n-3}\phi_1^{\mathbf{N}} \wedge \varepsilon = \tfrac{1}{5}\phi_1^{\mathbf{N}a}\varepsilon^b\zeta^i \wedge \phi_1^{\mathbf{N}c}\varepsilon^d\zeta^j \in \tfrac{1}{5} \bigwedge^2 V_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5 \subseteq W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5.$$

We consider now the second summand in (13), $5^{n-1}\mathbf{N}'\phi_1 \wedge u$. Since $\mathbf{N}' = (1-\sigma)^3 z$ for some unit z in $\mathbb{Z}_5[G]$, it suffices to show

$$5^{n-1}(1-\sigma)^3\, \phi_1 \wedge u \in W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5.$$

By assumption, $\psi_2(\phi_1) \equiv 0$ mod λ, and since $n \geqslant 2$, (11) shows $\psi_2(u) \equiv 0$ mod λ. Hence $\psi_2(\phi_1^{1-\sigma}) \equiv \psi_2(u^{1-\sigma}) \equiv 0$ mod λ^2. Since $\mathbb{Z}_5[\lambda]$ is a discrete valuation ring, and 5 differs by a unit from λ^4 in $\mathbb{Z}_5[\lambda]$, there exist $a, b, c, d \in \mathbb{Z}_5[\lambda]$ with $ad - bc = 5^{n-1}\lambda$, such that if A, B, C, D are corresponding lifts to $\mathbb{Z}_5[G]$, then $\psi_2(\phi_1^{(1-\sigma)A}u^{(1-\sigma)B}) = 0$, and $\psi_2(\phi_1^{(1-\sigma)C}u^{(1-\sigma)D}) \equiv 0$ mod $\lambda^{4(n-1)+3}$, and

$$5^{n-1}(1-\sigma)^3\, \phi_1 \wedge u = \phi_1^{(1-\sigma)A}u^{(1-\sigma)B} \wedge \phi_1^{(1-\sigma)C}u^{(1-\sigma)D}.$$

Since ψ_1 of any multiple of $1-\sigma$ vanishes, there is an integer k such that

$$\psi(\phi_1^{(1-\sigma)A}u^{(1-\sigma)B}\zeta^k) = 0.$$

Note therefore that

$$((1-N\mathfrak{q})/5^n)\, \phi_1^{(1-\sigma)A}u^{(1-\sigma)B}\zeta^k \in U_{S,\{\mathfrak{q}\}}$$

and that $((1-N\mathfrak{q})/5^n)\, \phi_1^{(1-\sigma)A}u^{(1-\sigma)B}\zeta^k$ is of norm 1. Since G is cyclic, and the prime above λ in K has as its decomposition group all of G, Lemma 3.3 of [R] gives us that $\#(H^1(G, U_{S,\{\mathfrak{q}\}})) \mid h_k = 1$. So $H^1(G, U_{S,\{\mathfrak{q}\}})$ vanishes, and there exists a $v \in U_{S,\{\mathfrak{q}\}}$ such that

$$v^{1-\sigma} = ((1-N\mathfrak{q})/5^n)\, \phi_1^{(1-\sigma)A}u^{(1-\sigma)B}\zeta^k.$$

Since $((1-N\mathfrak{q})/5^n)$ is a unit in $\mathbb{Z}_5$, it is now enough to show that

$$v^{1-\sigma} \wedge \phi_1^{(1-\sigma)C}u^{(1-\sigma)D} = v \wedge \phi_1^{(1-\sigma)^2C}u^{(1-\sigma)^2D} \in W_{S,\{\mathfrak{q}\}} \otimes \mathbb{Z}_5.$$

But we now have that $\psi_2(\phi_1^{(1-\sigma)^2 C} u^{(1-\sigma)^2 D}) = 0$, and again, ψ_1 of any multiple of $1-\sigma$ vanishes, so there is an l such that

$$\psi(\phi_1^{(1-\sigma)^2 C} u^{(1-\sigma)^2 D} \zeta^l) = 0,$$

which is what we had to show.

We now assume the theorem for T, and show that it then holds for any $T \cup \{\mathfrak{p}\}$. So let $\varepsilon_{S,T} \in W_{S,T}$ be such that

$$R_\eta(\varepsilon_{S,T}) = \theta_{S,T}^{(2)}(0).$$

Since

$$(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, \theta_{S,T}^{(2)}(0) = \theta_{S,T \cup \{\mathfrak{p}\}}^{(2)}(0),$$

is suffices to prove that $(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, \varepsilon_{S,T} \in W_{S, T \cup \{\mathfrak{p}\}}$. This is a consequence of the following lemma.

Lemma 3. $(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, W_{S,T} \subseteq W_{S, T \cup \{\mathfrak{p}\}}$.

Proof. First we will show that

$$(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p}) \textstyle\bigwedge^2 V_{S,T} \subseteq \bigwedge^2 V_{S, T \cup \{\mathfrak{p}\}}.$$

To do this, it suffices to check that $u_1, u_2 \in V_{S,T}$ implies that $(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, u_1 \wedge u_2 \in \bigwedge^2 V_{S, T \cup \{\mathfrak{p}\}}$. Since $(\mathcal{O}_k/\mathfrak{p})^\times$ is cyclic, there exists integers a, b, c, d with $ad - bc = 1$ and such that $u_1^a u_2^b \equiv 1 \bmod \mathfrak{p}$. Then

$$(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, u_1 \wedge u_2 = u_1^a u_2^b \wedge (u_1^c u_2^d)^{1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p}} \in \textstyle\bigwedge^2 V_{S, T \cup \{\mathfrak{p}\}}.$$

It suffices now to show that

$$(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p}) \textstyle\bigwedge^2 U_{S,T} \subseteq \bigwedge^2 U_{S, T \cup \{\mathfrak{p}\}}.$$

First suppose that $\mathfrak{p}$ stays prime in K. Take $u_1, u_2 \in U_{S,T}$. Then as before, $(\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)^\times$ is cyclic, so there exists $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, and $u_1^a u_2^b \equiv 1 \bmod \mathfrak{p}\mathcal{O}_K$, so

$$(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, u_1 \wedge u_2 = u_1^a u_2^b \wedge (u_1^c u_2^d)^{1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p}} \in \textstyle\bigwedge^2 U_{S, T \cup \{\mathfrak{p}\}}.$$

From now on we assume that $\mathfrak{p}$ splits in K. It suffices to show for u_1, $u_2 \in U_{S,T}$, that $(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} N\mathfrak{p})\, u_1 \wedge u_2 = (1 - N\mathfrak{p})\, u_1 \wedge u_2 \in U_{S, T \cup \{\mathfrak{p}\}}$. As in the base case, for this it suffices to show that $(1 - N\mathfrak{p})\, u_1 \wedge u_2 \in U_{S, T \cup \{\mathfrak{p}\}} \otimes \mathbb{Z}_p$ for all primes p. As before, for $p \neq 5$ we can find a, b, c, $d \in \mathbb{Z}_p[G]$ with $u_1^a u_2^b \in U_{S, T \cup \{\mathfrak{p}\}} \otimes \mathbb{Z}_p$ and $ad - bc = 1$. Then one just notes that $(1 - N\mathfrak{p})\, u_1^c u_2^d \in U_{S, T \cup \{\mathfrak{p}\}} \otimes \mathbb{Z}_p$. When $p = 5$, we want to show that $5^n u_1 \wedge u_2 \in U_{S,\, T \cup \{\mathfrak{p}\}} \otimes \mathbb{Z}_5$ where $5^n \parallel N\mathfrak{p} - 1$. Let ψ be the composite $U_{S,\, T} \to (O_K/\mathfrak{p})^\times \to (O_K/\mathfrak{p})^\times \otimes \mathbb{Z}_5 \cong \mathbb{Z}/5^n\mathbb{Z}[G]$. In $\mathbb{Z}/5\mathbb{Z}[G]$, for any given pair of elements, one divides the other, so we can find a, b, c, $d \in \mathbb{Z}_5[G]$ so that $ad - bc = 1$, and so that $\psi(u_1^a u_2^b) = 0 \bmod 5$. Hence if $u_1' = u_1^a u_2^b$, $u_2' = 5 u_1^c u_2^d$, then $5^n u_1 \wedge u_2 = 5^{n-1} u_1' \wedge u_2'$, with $\psi(u_1') = \psi(u_2') = 0 \bmod 5$. Identifying $5(\mathbb{Z}/5^n\mathbb{Z}[G])$ with $\mathbb{Z}/5^{n-1}\mathbb{Z}[G]$ and continuing inductively, we get u_1'', u_2'' with $\psi(u_1'') = \psi(u_2'') = 0 \bmod 5^n$, such that $5^n u_1 \wedge u_2 = u_1'' \wedge u_2''$, as desired.

APPENDIX

The purpose of this appendix is to describe some of the geometry underlying the claims in Section 3. First of all, we embed $C \to J$ via the divisor class map $\mathscr{P} \to cl(\mathscr{P} - \infty)$, and denote its image by Θ, a theta divisor. For a point $v \in J$, we let Θ_v denote the image of Θ under the translation-by-v map. We let O denote the origin on J.

Recall that J is birational to the symmetric product $C^{(2)}$, and if $\mathscr{P}_i = (x_i, y_i)$, $i = 1, 2$, are independent generic points on C, then all functions on J can be written as symmetric functions of x_i and y_i. In particular, we take

$$v = -\zeta x_i x_2,$$

$$\xi = \frac{1}{2\zeta^2 + \zeta + 2} \; \frac{(x_1 + x_2)(x_1 x_2)^2 + 1/2 - 2 y_1 y_2}{(x_1 - x_2)^2}.$$

Let $U = [\lambda] Q$. It can be shown using the group law from [Gra1] that the functions v and ξ were concocted to have divisors

$$(v) = \Theta_P + \Theta_{-P} - 2\Theta \qquad \text{and} \qquad (\xi) = \Theta_U + \Theta_{-U} - 2\Theta.$$

Let $\mathfrak{p}$ be a prime of $\mathcal{O}_k$ at which our model for C has good reduction, so $\mathfrak{p}$ is any prime other than λ or 2. It is shown in [BGL] that the intersection of $\Theta \cap J[5] \bmod \mathfrak{p}$ consists of O, $\pm P$, $\pm [\zeta^i] Q$, $0 \leqslant i \leqslant 4$, so R cannot lie on the support of (v) or (ξ), which is why we chose v and ξ as candidates for building units from $J[5]$.

A long calculation using the techniques of [Gra1] gives the minimal polynomial of $v(R)$ over k, and the methods also allow one to calculate

$$\begin{aligned}\sigma(v(R)) = v(R + P) = \tfrac{1}{5}(&5(\zeta^2 - \zeta^3)\, v(R)^4 + (-2 + \zeta + 19\zeta^2 - 28\zeta^3)\, v(R)^3 \\ &+ (-11 + 3\zeta - 33\zeta^2 - 14\zeta^3)\, v(R)^2 \\ &+ (-6 - 12\zeta + 12\zeta^2 - 19\zeta^3)\, v(R) - 12 + \zeta - 16\zeta^2 - 8\zeta^3).\end{aligned}$$

Similarly, one gets

$$\xi(R)=\frac{1}{2\zeta^2+\zeta+2}\,(-\zeta v(R)^4-(1+6\zeta)\,v(R)^3$$
$$-5(1+\zeta)\,v(R)^2+(1-3\zeta-\zeta^3)\,v(R)-3-3\zeta-\zeta^2-2\zeta^3),$$

so knowing $v(R+[i]P)$, $0\leqslant i\leqslant 4$, in terms of $v(R)$, we can calculate the minimal polynomial of $\xi(R)$ given in (3), and the action of σ on $\xi(R)$.

We will now give a proof of the proposition. Let

$$\Psi(z)=\frac{v(z)^2\,v(z+P)}{v(z+[3]P)\,v(z+[4]P)^2},$$

so that $\Psi(R)=v(R)^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)}$. Then the divisor of Ψ is $5(\Theta_P-\Theta)$. The results of [Gra1] show that $(y_1-1/2)(y_2-1/2)$ has the same divisor, so there is a constant κ such that

$$\frac{v(z)^2\,v(z+P)}{v(z+[3]P)\,v(z+[4]P)^2}=\kappa(y_1-1/2)(y_2-1/2)(z). \tag{14}$$

Multiplying (14) by (14) with z replaced by $-z$, we get

$$\frac{1}{\prod_{i=0}^{4}\,v(z+[i]P)}=-\kappa^2, \tag{15}$$

since $(y_1-1/2)(y_2-1/2)(y_1+1/2)(y_2+1/2)=-v^5$, and v is an even function. Plugging in $z=R$ into (15), we get that $-\kappa^2$ is the negative of the constant term of the minimum polynomial of $v(R)$ over k, which by (2) is -1. Therefore $\kappa=\pm 1$. From Lemma 10 in [Gra3], there is a function $F\in k(J)$ such that $(y_1-1/2)(y_2-1/2)([\lambda]z)=F(z)^5$, so we have

$$\frac{v([\lambda]z)^2\,v([\lambda]z+P)}{v([\lambda]z+[3]P)\,v([\lambda]z+[4]P)^2}=\pm F(z)^5. \tag{16}$$

In particular, if we let $E\in J[\lambda^5]$ be such that $[\lambda]E=R$, then letting $z=E$ in (16) we have

$$v(R)^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)}=\pm F(E)^5.$$

To prove the proposition therefore, we need to show that $\zeta^{-1}F(E)^5$ is a fifth power in K, or equivalently, that $\zeta^{-1/5}F(E)$ is in K. To do this, it suffices to show for every prime $\varpi\neq\lambda$, that if its Frobenius Fr_ϖ in $\mathrm{Gal}(k(\zeta^{1/5}, J[\lambda^5])/k)$ fixes $K=k(\varepsilon^{1/5})$, then it also fixes $\zeta^{-1/5}F(E)$. Every such ϖ can be taken so that $\varpi \equiv 1 + a\lambda^3 + b\lambda^4 \bmod \lambda^5$, $a, b \in \mathbb{Z}/5\mathbb{Z}$. First recall that the complementary laws of quintic reciprocity [Gra3] show that

$$\mathrm{Fr}_{\varpi}(\zeta^{1/5}) = \zeta^{a+b}\zeta^{1/5},$$

$$\mathrm{Fr}_{\varpi}(\varepsilon^{1/5}) = \zeta^{a}\varepsilon^{1/5},$$

so the Frobeniuses which fix K are precisely the Fr_{ϖ} with $a = 0$. Now recall from [Gra3] that the theory of complex multiplication says that the action of Fr_{ϖ} on E is induced by $[\varpi\varpi_3]$, where $\varpi_3 = \rho_3(\varpi)$, and $\rho_i \in \mathrm{Gal}(k/\mathbb{Q})$ satisfies $\rho_i(\zeta) = \zeta^i$. Since $\varpi\varpi_3 \equiv 1 + 2b\lambda^4 \bmod \lambda^5$, we have

$$\mathrm{Fr}_{\varpi}(E) = E + [2b]P.$$

It is shown in [Gra3] that $F(z+P) = \zeta^3 F(z)$, so

$$\mathrm{Fr}_{\varpi}(\zeta^{-1/5}F(E)) = \zeta^{-b}\zeta^{-1/5}F(E+[2b]P) = \zeta^{-b}\zeta^{-1/5}\zeta^{6b}F(E) = \zeta^{-1/5}F(E),$$

as desired.

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